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Relativistic diffusion model

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Abstract. Evidence is presented that diffusion drives colliding many-particle systems at relativistic energies from the initial δ–functions in rapidity towards the equilibrium distribution. Analytical solutions of a linear Fokker-Planck equation represent rapidity spectra for participant protons in central heavy-ion collisions at SPS-energies accurately. Thermal equilibrium in the interaction region is not attained, nonequilibrium features persist and can account for the broad rapidity spectra.

PACS. 25.75-q Relativistic heavy-ion collisions – 24.60.Ky Fluctuation phenomena – 24.10.Jv Relativistic models

Relativistic heavy-ion collisions [1] offer a unique possibility to study equilibration processes in microscopic many-body systems. In particular, rapidity distributions for participant protons are valuable indicators for the time-evolution of the reactions. They can be used to investigate the approach to thermal equilibrium in the interaction region.

Isotropic thermal models fail to reproduce the broad, often flat or even dipped rapidity spectra that have been observed at AGS and SPS energies. Hence, these models have been supplemented [2] by collective expansion and anisotropic flow schemes, or by a superposition of isotropically decaying thermal sources (fireballs) [3] to fit the data [4].

In this note I introduce a nonequilibrium-statistical model based on a linear Fokker-Planck equation (FPE) [5] to describe the gradual approach to local thermal equilibrium in the interaction zone. It is argued that the broad rapidity distributions for participant protons that have been observed may also indicate nonequilibrium properties of the system: Due to the short interaction time, thermal equilibrium in the interaction region is not reached even in case of central collisions of heavy systems such as Pb+Pb.

The approach is macroscopic: It is concerned with the time-evolution of rapidity distribution functions for participant protons towards statistical equilibrium. This relaxation process occurs as a consequence of random collisions and creation of secondaries (about 2100 pions in a central Pb+Pb-collision, with an estimated 40% produced directly). The nonequilibrium rapidity spectrum for protons emerges analytically from the model. A similar approach has previously been used to evaluate and predict transverse energy distributions (integrated over all particle species) analytically [6]. The present model allows, in particular, to interpret flat or even double-peaked proton rapidity spectra that have been observed in heavy relativistic systems [7–10] as being due to incomplete thermalization.

In the relativistic diffusion model for rapidity spectra of participants, the initial projectile rapidity

$$y_1 = \frac{1}{2} \ln \left(\frac{E_L + P_L}{E_L - P_L} \right) \tag{1}$$

with laboratory energy E_L and momentum P_L, as well as the initial target rapidity y_2 (= 0 for fixed-target experiments) relaxes towards the equilibrium value y_{eq} given by the center-of-mass rapidity due to collisions, particle creations and decays.

The discussion is restricted to participant protons which (as compared to produced particles) exhibit the nonequilibrium properties most clearly. Transforming to

$$Y = \frac{2y}{y_1} - 1 \tag{2}$$

such that Y = 1 for y = y_1, and Y = -1 for y = 0, the initial conditions of the time-dependent rapidity distribution functions for projectile (target) participants are $R_{1,2}(Y,0) = \delta(Y \mp 1)$. Here I neglect a small initial broadening $Y \approx 0.07$ which is due to the Fermi momentum.

A partial differential equation for the distribution function R (Y,t) of the macroscopic variable Y should account for both the shift of the mean value $\langle Y \rangle$ towards equilibrium Y_{eq} and the corresponding broadening in time. The Fokker-Planck equation successfully describes similar nonequilibrium processes in many areas of physics [5] –

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starting with the theory of Brownian motion, where it has been used to model both the velocity- and the coordinate distribution. The use of the FPE is strictly valid only near thermal equilibrium. However, the collisions investigated here are central reactions with relatively long interaction times; they correspond to few percent of the cross section. Although the analysis will show that thermal equilibrium is apparently not reached in these collisions, one comes sufficiently close to it to justify use of the FPE. In relativistic systems, however, Lorentz-invariant kinematical variables have to be introduced. In particular, the rapidity replaces the velocity to describe the motion parallel to the beam direction, and the FPE for the probability distribution of the rapidity becomes

$$\frac{\partial R(Y,t)}{\partial t} = \frac{1}{\tau_Y}\frac{\partial}{\partial Y}\Big[(Y-Y_{eq})R(Y,t)\Big] + D_Y\frac{\partial^2}{\partial Y^2}R(Y,t). \quad (3)$$

Two transport coefficients determine the time evolution of the rapidity distribution, the relaxation time τ_Y and the rapidity diffusion coefficient D_Y. Rapidity spectra calculated from (3) will ultimately depend on two dimensionless parameters: τ_{int}/τ_Y with the interaction time τ_{int} (the final time in the solution of (3)), and $\sqrt{D_Y\tau_Y}$. The first parameter describes how close to equilibrium the system is, and the second parameter determines the width of the rapidity distribution at equilibrium. Both may be fitted from experimental rapidity distributions.

In this work, the interaction time will be taken from other sources [6]. The relaxation time τ_Y and hence, the ratio of these two times will be fitted to data. The rapidity relaxation time is related to the diffusion coefficient through the temperature of the thermal equilibrium distribution that is obtained from other origins such as hadron abundancies. The corresponding dissipation-fluctuation theorem may, however, be violated in relativistic collisions where a large amount of energy resides in the interaction. Hence, I will consider the transport coefficients as independent parameters when comparing with data.

The relaxation time determines the speed of the approach to the Gaussian stationary solution given by $\partial R_{eq}/\partial t = 0$, with the corresponding equation for the mean value

$$\frac{d}{dt}\langle Y(t)\rangle = \frac{-1}{\tau_Y}[\langle Y(t)\rangle - Y_{eq}]. \quad (4)$$

Here, τ_Y may be considered as a macroscopic quantity and – based on a suitable parametrization – determined in a fit to available data, or derived in a microscopic model. In this work I treat it on the macroscopic level. Hence, the model is positioned between the available microscopic models of BUU-type on the one hand, and macroscopic thermal models on the other hand. It maintains advantages of thermal models such as transparency, analytical treatment, and few intuitive parameters, but simultaneously goes beyond the thermalization assumption and allows to investigate nonequilibrium effects.

The diffusion coefficient D_Y is responsible for the associated broadening of the distribution functions due to collisions, particle creations and decays. For a linear FPE, the fluctuations are usually determined without detailed consideration of the microscopic structure of the system, because the stationary solution R_{eq} of the FPE should agree with the isotropic thermal equilibrium distribution R_{th}. The latter depends on a temperature T_p that is to be identified with the freeze-out temperature, and on the particle mass m_p. In the Boltzmann approximation, this thermal distribution is expressed as [2]

$$R_{th}(y_{cm}) = c_p^T m_p^2 T_p\Big[1+2\chi_T(y_{cm})+2\chi_T(y_{cm})^2\Big] \times \exp\left(\frac{-1}{\chi_T(y_{cm})}\right) \quad (5)$$

with the center-of-mass rapidity $y_{cm} = y_1 Y/2$, and

$$\chi_T(y) = \frac{T_p}{m_p\cosh(y)}. \quad (6)$$

The freezeout-temperature T_p can be approximately determined from Boltzmann fits to transverse mass spectra or hadron yields [2],[4], [7–10]. The constant c_p^T ensures that the distribution function in Y-space is normalized to 1 for each temperature T_p.

In thermal models of relativistic collisions [2], it is assumed that the system attains the thermal equilibrium distribution (5) and subsequently expands collectively. On the basis of the present nonequilibrium-statistical approach, however, it appears that due to the short interaction time, the system does not reach the equilibrium distribution at SPS-energies, such that broad rapidity spectra could occur due to nonequilibrium properties. A possible collective expansion which might occur on top of this would require a separate treatment, it cannot be described within the FPE-framework alone.

The shape of the thermal distribution $R_{th}(Y)$ is not exactly Gaussian, although very close to it. Hence, it cannot be equated directly with the stationary solution $R_{eq}(Y)$ of the FPE as in usual nonequilibrium-statistical approaches. Instead, a Gaussian approximation to the thermal distribution is used to equate it with the stationary solution of the FPE. I obtain the rapidity diffusion coefficient as

$$D_Y^{th} = \frac{1}{2\pi\tau_Y}\left[c_p^T m_p^2 T_p\left(1+2\frac{T_p}{m_p}+2\left(\frac{T_p}{m_p}\right)^2\right)\right]^{-2} \times \exp\left(\frac{2m_p}{T_p}\right). \quad (7)$$

With this choice of the diffusion coefficient, thermal and stationary distribution become almost identical. For a given equilibrium distribution, the rapidity diffusion coefficient is then solely determined by the dissipative constant τ_Y. This equation is reminiscent of the well-known Einstein relation between diffusion and friction coefficient in the analysis of Brownian motion [11]. Due to the temperature-dependence of the normalization constant c_p^T, it displays the expected behavior, namely, the diffusion coefficient at fixed incident rapidity y_1 rises almost linearly with increasing temperature of the corresponding equilibrium distribution.

In relativistic collisions, however, the above dissipation-fluctuation theorem need not be fulfilled because a large amount of energy resides in the interaction, and is used for the generation of secondary particles. Hence, in a comparison with data, values of the rapidity diffusion coefficient exceeding the ones obtained from (7) may be expected, and are indeed found at SPS-energies. Consequently, I treat the rapidity relaxation times and the diffusion coefficients as independent parameters at these energies.

For participant protons, δ-function initial conditions lead to superposed Gaussian solutions of the FPE with mean values

$$\langle Y_{1,2}(t)\rangle = Y_{eq}\left(1-\exp\left(-\frac{t}{\tau_Y}\right)\right) \pm \exp\left(\frac{-t}{\tau_Y}\right) \quad (8)$$

and variances (for each of the projectile- and target-like distributions)

$$\sigma_Y(t)^2 = D_Y\tau_Y\left(1-\exp\left(\frac{-2t}{\tau_Y}\right)\right). \quad (9)$$

For small times, two separate Gaussians develop near target and projectile rapidity. They shift towards the equilibrium value and simultaneously broaden with increasing time due to collisions and particle creations. For $t/\tau_Y \geq 0.2$ at SPS-energies, the superposed distribution that differs from a Gaussian develops. For $t/\tau_Y \geq 1.1-1.5$ (depending on the asymmetry and size of the systems) the superposed distribution acquires a Gaussian shape, and its width shrinks until the stationary distribution ($\Gamma_{eq} = \sqrt{8\ln 2}\sqrt{D_Y\tau_Y}$) is reached for $t \to \infty$. Due to the finite values of the interaction times (*incomplete stopping*), actual rapidity distribution functions remain significantly broader than the stationary distribution and differ from Gaussians.

Values of the interaction times $t = \tau_{int}(b)$ that are – together with the corresponding energy relaxation time τ – in accordance with measured transverse energy distributions may be taken from [6]. Model-dependent HBT-analyses of effective source lifetimes and also microscopic calculations yield smaller values [1] which are closer to the transit time [6]. However, since diffusion-model predictions depend on τ_{int}/τ for the energy relaxation [6], and on τ_{int}/τ_Y for the rapidity relaxation of participants, the use of smaller values for the interaction times will simply lead to a rescaling of the relaxation times and diffusion coefficients, whereas the dimensionless parameters τ_{int}/τ_Y and $\sqrt{D_Y\tau_Y}$ (and hence, the physical conclusions) will not be changed.

Figure 1 shows distributions R (Y,t) weighted with the respective numbers of projectile and target proton participants $N_{p1,2}(b=0)$ at several selected values of t/τ_Y for the asymmetric S + Au system at SPS energies. Both the asymmetric relaxation in rapidity space and the approach to the stationary distribution at large times are clearly displayed. For a symmetric system such as Pb + Pb (Fig. 2), the relaxation is, of course, completely symmetric in Y.

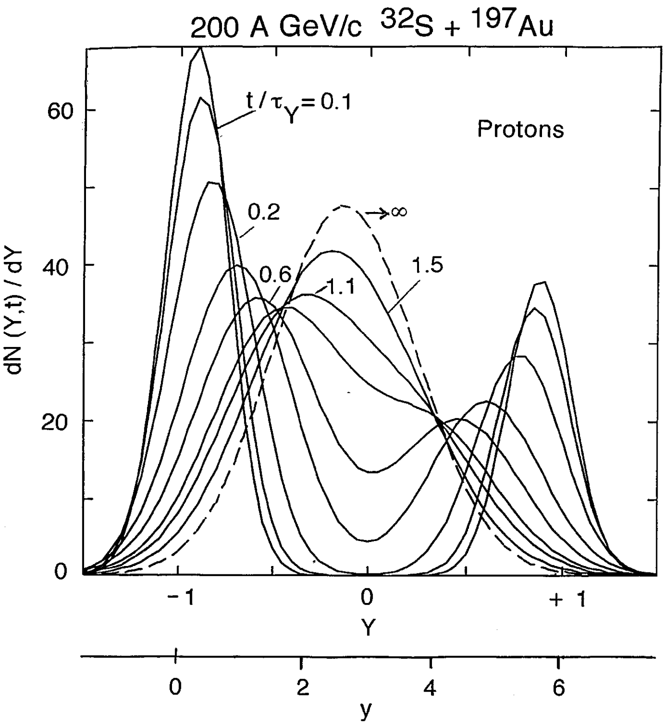

Fig. 1. Rapidity relaxation of participant protons in 200 A GeV/c S + Au. Selected analytical solutions of the Fokker-Planck equation for various values of time t/τ_Y weighted with the respective numbers of projectile and target participants for central collisions illustrate the asymmetric equilibration. The dashed curve is the Gaussian stationary distribution (centered at $y_{eq} = 2.63$, $Y_{eq} = -0.13$). For comparison with NA35 data cf. Fig. 3

To compute actual rapidity density distributions for participant protons in the relativistic diffusion model, one needs to know $N_{p1,2}(b)$ at impact parameter b, the impact-paramter dependent interaction times $t = \tau_{int}(b)$, and the cross-section to perform the integral over b. The impact-parameter dependent rapidity spectrum (that may be integrated over b) is obtained as

$$\begin{aligned}\frac{dn}{dy}|_b &= \frac{2}{y_1}\frac{dN}{dY}|_b \\ &= \frac{2}{y_1}\Big[N_{p1}(b)R_1(Y,t=\tau_{int}(b)) \\ &\quad + N_{p2}(b)R_2(Y,t=\tau_{int}(b))\Big]. \quad (10)\end{aligned}$$

For central collisions with small cross sections (typically 5% of the total cross section at SPS-energies), it is sufficient to evaluate this expressions at b=0 since the spectra do not depend much on the precise magnitude of the cross section.

In order to illustrate the rapidity relaxation for protons at SPS-energies, I have investigated the systems S+Au, S+S and Pb+Pb where data are now available [7–10]. For

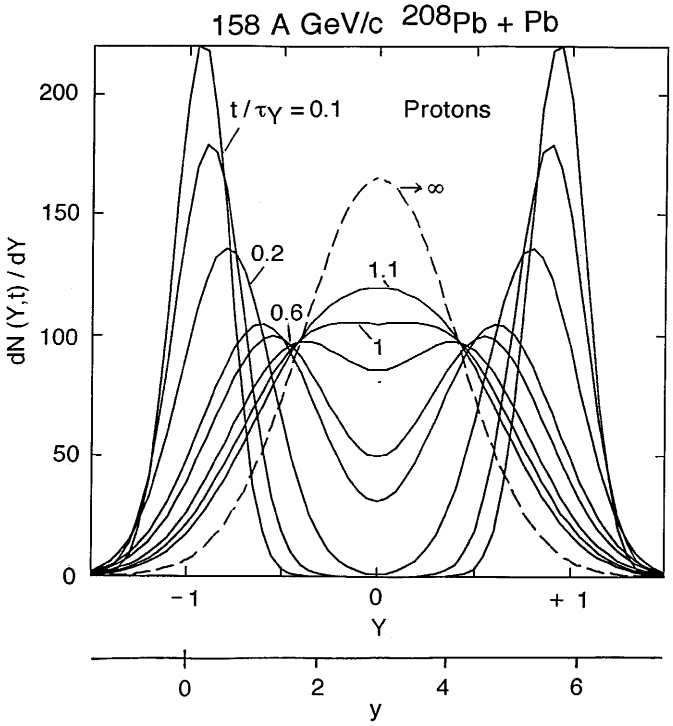

Fig. 2. Rapidity relaxation of participant protons in 158 A GeV/c Pb + Pb. Selected analytical solutions for various values of time t/τ_Y as in Fig. 1 are shown. The stationary distribution (dashed) is centered at midrapidity ($y_{eq} = 2.91, Y_{eq} = 0$). For comparison with NA49 central collision data cf. Fig. 3

symmetric systems the net proton yield ($p-\overline{p}$) can be determined fairly accurately from the charge excess distribution. Protons from the decay of hyperons are not included in this comparison with the diffusion model. (Similar to rapidity distributions of produced mesons, these secondary particles are peaked at mid-rapidity.) Hence, the integrals of the rapidity spectra tend to be smaller than the number of proton participants for central collisions.

Rapidity density distributions for protons in the asymmetric S+Au system at SPS-momentum of 200 A GeV [8,9] are compared with the diffusion-model results in the upper part of Fig. 3. The diffusion coefficient that fits the data (Table 1) is a factor of 12 larger than the value calculated from the fluctuation-dissipation theorem (7). In all the systems investigated at SPS-energies, such a large diffusion coefficient is impossible to achieve through realistic variations of the temperature and hence, the Einstein relations are clearly violated. Further work tends to show, however, that the theorem (7) is fulfilled at lower (SIS-) energies [16]. The origin of the violation at higher energies will need further investigation. It may certainly also mask collective expansion.

For S+Au, the model reflects the asymmetry that is present in the data. The experimental difficulties to separate participants from spectators in asymmetric systems [8] may cause part of the deviation of the midrapidity-datapoint. Equilibrium models fail to generate such asymmetric distributions.

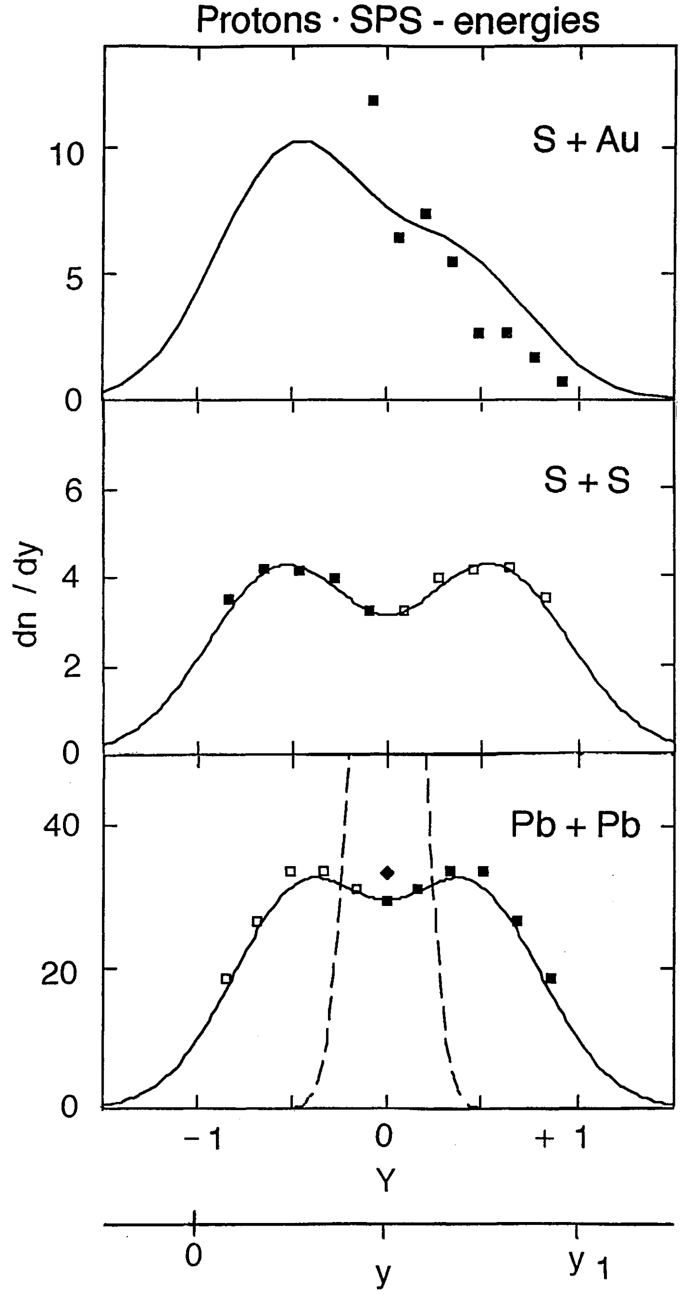

Fig. 3. Calculated rapidity density spectra of participant protons in central S+Au, S+S and Pb+Pb- collisions at SPS- energies in comparison with NA 35 [7,8] and NA 49 [10] data (black squares; data for symmetric systems reflected at $y_{eq} = y_1/2$). The projectile rapidities are $y_1 = 6.06$ for S, and $y_1 = 5.83$ for Pb. For Pb+Pb the thermal equilibrium distribution (T=160 MeV) is also shown; its maximum is at dn/dy = 165 (dashed). The data are incompatible with thermal distributions. The time-of- flight data point (diamond) includes secondary protons from hyperon decays, the other (preliminary) data points do not. The integrals of the respective theoretical distributions correspond to 45, 25 (renormalized) and 164 protons. As a consequence of incomplete y-relaxation at SPS- energies, the spectra are significantly broader than the equilibrium distributions, and their shapes differ from Gaussians. This emphasizes the non-equilibrium nature of relativistic collisions

Table 1. Relaxation times τ_Y and rapidity diffusion coefficients D_Y for systems at SPS-energies. The dimensionless parameter τ_{int}/τ_Y determines how close to equilibrium the system is, whereas $\sqrt{D_Y \tau_Y}$ gives the standard deviation of the rapidity distribution at equilibrium

System	p_L (GeV)	y_1	τ_Y (fm)	D_Y (10^{-4}fm^{-1})	τ_{int}/τ_Y	$\sqrt{D_Y \tau_Y}$
S + Au	200	6.06	72	26	0.84	0.374
S + S	200	6.06	25	89	0.60	0.451
Pb + Pb	157.7	5.83	134	12	0.81	0.408

Using the transport coefficients (or dimensionless parameters) of Table 1, the broad double-peaked proton rapidity spectrum shown in the middle part of Fig. 3 for S+S is in reasonable agreement with the experimental results [7,8]. In particular, the positions of the maxima of the experimental distributions are precisely reproduced for $\tau_{int}/\tau_Y = 0.6$. The occurence of these maxima indicates that the process is far from equilibrium: An equilibrium distribution has no dip at central rapidity. Impact-parameter integration will sligthly smooth out the distribution. Thermal descriptions (without or with flow) cannot reproduce the data. Again, the Einstein relations are violated by a factor of about 12 because a large amount of energy resides in the interaction and is used for particle production.

For Pb+Pb at SPS-momentum of 158 GeV per particle the situation is similar. The net proton ($p - \overline{p}$) preliminary data points that have been determined from the charge excess distribution of hadrons [10] are in agreement with the diffusion-model result for $\tau_{int}/\tau_Y = 0.81$, and a diffusion coefficient that exceeds the one calculated from the dissipation-fluctuation theorem by a factor of 9. The time-of-flight datapoint (diamond) includes secondary protons from hyperon decays, which are not contained in the model. Refined data analysis of the net proton distribution indicates [9] that the dip may turn into a flat distribution. This would require $\tau_{int}/\tau_Y \geq 0.9$ in the relativistic diffusion model, cf. Fig. 2.

The thermal distribution for T = 160 MeV is also shown, dashed curve. It clearly disagrees with the data. The temperatures of the thermal distributions are taken to be consistent with previous analyses of transverse momentum distributions and hadron yields [4], [8], [12]. For symmetric systems and diffusion coefficients from (6), these distributions coincide with the stationary solutions of the FPE.

The transport coefficients that determine the rapidity spectra are given in Table 1. The dependence of the rapidity relaxation time τ_Y on size and asymmetry of the participant systems as well as on the available energies is in reasonable agreement with the corresponding expression for the energy relaxation time τ that was found in [6] and hence, this can be used for predictions. For the systems investigated here, the rapidity relaxation times are larger than the interaction times for central collisions, which emphasizes the nonequilibrium behaviour. The ratio to the energy relaxation time is $\tau_Y/\tau \approx 1.3$, indicating that the rapidity relaxation of participants is slower than the buildup of transverse energy. The rapidity diffusion coefficient is about 7 times larger for the small S+S system than for Pb + Pb, essentially because of its inverse proportionality to the relaxation time. The transport coefficients given here properly describe the broad nonequilibrium rapidity spectra for participant protons. Note, however, that the broad experimental distributions may also have other additional sources such as collective expansion which is not accounted for in the diffusion approach.

Lighter produced particles such as pions and kaons have fairly wide thermal distributions in rapidity due to their small masses. They are expected to be less sensitive to the nonthermal behaviour that the relativistic diffusion model describes. It will, however, tend to broaden their rapidity spectra and has to be considered in precise analyses of the data.

Considering proton rapidity spectra at the lower AGS-momenta of 11–15 A GeV [12,13], the mean interaction times are larger [6] and hence, one may expect to come closer to thermal equilibrium. Still, the relatively small Si+Al system exhibits in central collisions a doubled-peaked proton rapidity distribution [13], and for Au+Au it is flat [12]. Both are significantly broader than isotropic thermal distributions, which has been considered as evidence for thermalization plus longitudinal flow [2]. It is expected that the relativistic diffusion model offers an alternative interpretation based on the assessment that thermal equilibrium is not reached for participant protons in the interaction region. Again, lighter produced particles will be less sensitive to the nonthermal behaviour, although it tends to broaden their rapidity spectra - an effect that is masked by present flow-analyses.

At even lower SIS-energies of 1–2 A GeV, the light Ni+Ni system exhibits proton and deuteron rapidity spectra that are wider than expected for an isotropically emitting thermal source, whereas pion distributions appear to be thermal [14]. Nonequilibrium properties are likely to cause the discrepancy.

Comparing with equilibration processes in heavy-ion collisions at non-relativistic energies [15], the present investigation underlines the unexpected result that relaxation phenomena in small quantum systems persist over a range in incident energy from about 5 MeV per particle well into the ultra-relativistic region, bridging energies that differ by more than four orders of magnitude.

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